Non-diagrammatic calculation of QCD one-loop β-function based on the renormalization group equation

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Abstract

Using the higher covariant derivatives regularization of gauge theories in the framework of the background field method, supplemented with one-loop Pauli-Villars regulator fields, we obtain a version of the renormalization group equation for the regulator fields, whose vacuum energy depends on the background gauge field. It is evaluated using an anomalous Ward-Takahashi identity, which is related to the rescaling anomaly of the auxiliary fields, obtained by the Fujikawa approach. In this way the anomalous origin of the one-loop β-function in QCD is clearly shown in terms of scaling of effective Lagrangians without the use of any Feynman diagram. The simplicity of the method is due to the preservation of the background and quantum gauge invariance in any step of the calculation.

Introduction

In an interesting paper Polchinski demonstrated that Wilson's decimation method [1] applied to continuum field theory is sufficient to provide proof of perturbative renormalization [2]. To obtain the exact renormalization group equation (ERGE) he used an obvious identity, which consists in setting the integral functional of a proper total derivative to zero, and the independence of the partition function on the scale. Such an idea corresponds to a reparameterization of the partition function since the total derivative emerges from a field redefinition [3, 4].

This derivation of ERGE has to be modified when external gauge fields are present. In such a case, as they could be field dependent, we can not discard the singular terms that appear when developing the total derivative before having studied their possible physical relevance. In fact, as was shown in ref. [6] in the context of $N = 1$ supersymmetric Yang-Mills theory, they assume the meaning of a vacuum energy and are responsible for the exact one-loop running of the holomorphic gauge coupling [7].

Here the analysis in [6, 8] is extended to non-supersymmetric gauge theories. In those papers the gauge invariant regularization proposed by Arkani-Hamed and Murayama [9] is

used, which consists in giving a big mass to the extra fields[1] of a finite theory with extended supersymmetry. In conventional gauge theories we clearly have to resort to a different regularization. The attractive properties of gauge invariance, non-perturbative meaning and applicability to chiral and supersymmetric models make the regularization proposed by Slavnov [11] interesting. It is a hybrid of higher covariant derivatives and Pauli-Villars (PV) regularizations. Nevertheless it has some inconsistencies, the main one being known as overlapping divergences. Although minor modifications of the original scheme are possible, which make the regularization self-consistent [13, 14], it is not yet known how to use it in the RG context.

In this paper a solution to this is offered at the one-loop level using the background field method when the regulator fields are the only to flow. The outcome is a version of RG equation close to the ones in [6, 8], from which we obtain the one-loop β-function of non-supersymmetric $SU(N)$ Yang-Mills theory without using Feynman diagrams. The simplicity of the calculation is a result of the preservation of the background and quantum gauge invariance.

The background field method is technically useful for calculating the vacuum energy of regulator fields, but it also introduces conceptual simplifications. For instance, using the regularization mentioned above in its framework, a formal transition to a covariant background gauge is not required to prove the gauge invariance of one-loop divergences. The invariance of the partition function under gauge transformations of the background field makes it evident.

The paper is structured as follows. Noting that the gauge field can be considered external for the calculation of the one-loop β-function in QED, we begin with the Abelian theory to show how our method works without taking into account complications which are due to the quantum fluctuations. In section 2, recording the regularized gauge invariant effective action of the non-Abelian theory, we emphasize the important points of the Slavnov regularization and the background field method for our approach. In section 3 the calculation of the one-loop β-function is performed when the gauge group is $SU(N)$. The conclusions are followed by two appendices, the first one reporting the derivation of an equation that we shall term 't Hooft's and the second one the calculation of the Jacobians used in the text.

1 One-loop β-function of QED

As a result of Ward's identity in QED, it is a well known fact that the charge renormalization originates solely from vacuum polarization. Then, at the one-loop level, the quantum fluctuations of the gauge field can be disregarded to achieve the β-function[2]. To regularize the vacuum polarization diagram in a way which does not break gauge invariance, we shall use the PV regularization [16], for which some details will be given in the next section. It is regularized introducing a massive PV spinor field of bosonic type into the Lagrangian.

[1] With extra field we mean a field of the finite theory which does not appear in the theory we are regulating.

[2] Incidentally, with the gauge field treated as external, the vacuum polarization is the only divergent diagram. In fact, indicating with N the number of vertices, the spinor cycles are divergent when $N < 5$, $N = 2$ being the maximum grade. The cycle with $N = 4$ corresponds to photon-photon scattering and has a potentially logarithmic divergence, but, as a consequence of gauge invariance, it is actually convergent. Finally, using Furry's theorem, we can discard the loop with $N = 3$ (for these topics see for instance refs. [15]).

Indicating with ψ and ψ_1 the physics and PV field respectively, the Euclidean generating functional regularized to the M_0 scale is

$$Z[J,A;M_0] = \int \mathcal{D}\Psi \; \exp\Big\{ -\frac{1}{4e_0^2}\int_x F_{\mu\nu}^2 + \int_x \bar\psi(i\,\not{D} - m_0)\psi + \int_x \bar\psi_1(i\,\not{D} - M_0)\psi_1$$
$$+ \int_x (\bar\chi\psi + \bar\psi\chi) + \int_x (\bar\chi_1\psi_1 + \bar\psi_1\chi_1)\Big\} \; . \tag{1}$$

As A_μ is a classical field, we have not considered gauge fixing terms. The dependence on x has been understood and

$$\Psi \doteq \{\Phi;\Phi^{\rm PV}\} \doteq \{\psi,\bar\psi;\psi_1,\bar\psi_1\} \; , \qquad \mathcal{D}\Psi \doteq \mathcal{D}\psi\mathcal{D}\bar\psi\mathcal{D}\psi_1\mathcal{D}\bar\psi_1 \; ,$$
$$J \doteq \{\mathcal{J};\mathcal{J}^{\rm PV}\} \doteq \{\bar\chi,\chi;\bar\chi_1,\chi_1\} \; , \qquad \int_x \doteq \int d^4x \; . \tag{2}$$

Now Wilson's idea of RG is applied. It consists in lowering the scale M_0 to M while determining the effective action which compensates for the loss of modes. As we shall show, the functional

$$Z[J,A;M,M_0] = \int \mathcal{D}\Psi \; \exp\Big\{ -M\int_x \bar\psi_1\psi_1 - S_{\rm eff}[A,\Psi,\mathcal{J};M,M_0] + \int_x f_M(\bar\chi_1\psi_1 + \bar\psi_1\chi_1)\Big\}$$
$$\equiv \int \mathcal{D}\Psi \; \exp(-S_{\rm tot}) \tag{3}$$

is equal to $Z[J,A;M_0]$, except for a tree level two-point function, provided that the effective action and f_M satisfy proper RG equations with the initial conditions

$$\lim_{M\to M_0} S_{\rm eff}[A,\Psi,\mathcal{J};M,M_0] = \frac{1}{4e_0^2}\int_x F_{\mu\nu}^2 - \int_x \bar\psi(i\,\not{D} - m_0)\psi$$
$$- \int_x \bar\psi_1 i\,\not{D}\psi_1 - \int_x (\bar\chi\psi + \bar\psi\chi) \; , \tag{4}$$

$$\lim_{M\to M_0} f_M = 1 \; . \tag{5}$$

From the RG point of view we have only renormalized the sources associated with the fields which flow. Even if f_M could be x-dependent in principle, it will be shown that in fact it only depends on the scale.

The flow equations are obtained from a Polchinski identity:

$$0 = \int_x M\frac{\partial}{\partial M}\left(\frac{1}{M}\right)\int \mathcal{D}\Psi \; \Big\{ \frac{\delta}{\delta\psi_1}\left(\psi_1 M + \frac{1}{2}\frac{\delta}{\delta\bar\psi_1}\right)$$
$$+ \frac{\delta}{\delta\bar\psi_1}\left(\bar\psi_1 M + \frac{1}{2}\frac{\delta}{\delta\psi_1}\right)\Big\} \exp(-S_{\rm tot}) \; . \tag{6}$$

This identity has been written directly in the x-space because the role of the cut-off function is played by $1/M$, which does not depend on the momenta. This is not true for conventional cut-off methods. Indicating with

$$\langle \mathcal{O} \rangle = \int \mathcal{D}\Psi \; \mathcal{O} \; \exp(-S_{\rm tot}) \tag{7}$$

the quantum average of an operator $\mathcal{O}$ in an external field A_μ and in presence of sources J, eq. (6) becomes

$$0 = \int_x \left\langle \frac{\delta \psi_1}{\delta \psi_1} + \frac{\delta \bar{\psi}_1}{\delta \bar{\psi}_1} - \frac{\delta S_{\rm tot}}{\delta \psi_1}\psi_1 - \bar{\psi}_1 \frac{\delta S_{\rm tot}}{\delta \bar{\psi}_1} + \frac{1}{M}\left(\frac{\delta S_{\rm tot}}{\delta \psi_1}\frac{\delta S_{\rm tot}}{\delta \bar{\psi}_1} - \frac{\delta^2 S_{\rm tot}}{\delta \psi_1 \delta \bar{\psi}_1}\right)\right\rangle . \tag{8}$$

We have come across the quantity

$$\left\langle \frac{\delta \psi_1}{\delta \psi_1} + \frac{\delta \bar{\psi}_1}{\delta \bar{\psi}_1} \right\rangle . \tag{9}$$

Similar terms were discarded in ref. [2] and in the following literature on RG with the exception of refs. [3]–[6], [8]. They have been interpreted as “Wilson lines biting their own tails” in the gauge invariant formulation of the exact RG proposed by Morris [3, 5]. Showing their anomalous origin when an external gauge field is present[3] and the flow of regulator fields with respect to a mass parameter is considered, the physical meaning of terms analogous to (9) has been elucidated in refs. [6, 8]. From this point of view it is clear why Polchinski could discard these terms: this is a legitimate assumption for a theory like $\lambda\phi^4$ that does not have a background gauge field[4]. Following [6, 8], we have to evaluate the quantity (9) carefully, as it assumes the meaning of vacuum energy of ψ_1 and $\bar{\psi}_1$ when the external gauge field A_μ is present. We shall calculate this quantity after the RG equations have been obtained.

Substituting $S_{\rm tot}$ in eq. (8) for the expression defined in (3), we obtain

$$\begin{aligned} 0 = \Bigg\langle & \frac{1}{2}\int_x \left(\frac{\delta \psi_1}{\delta \psi_1} + \frac{\delta \bar{\psi}_1}{\delta \bar{\psi}_1} \right) + \frac{1}{M}\int_x \left(\frac{\delta S_{\rm eff}}{\delta \psi_1}\frac{\delta S_{\rm eff}}{\delta \bar{\psi}_1} - \frac{\delta^2 S_{\rm eff}}{\delta \psi_1 \delta \bar{\psi}_1}\right) \\ & -M\int_x \bar{\psi}_1\psi_1 + \int_x f_M(\bar{\chi}_1\psi_1 + \bar{\psi}_1\chi_1) - \frac{1}{M}\int_x f_M^2 \bar{\chi}_1\chi_1 \Bigg\rangle . \end{aligned} \tag{10}$$

Note that we have used

$$\frac{\delta^2 S_{\rm tot}}{\delta \psi_1 \delta \bar{\psi}_1} = \frac{1}{2}\left(\frac{\delta^2 S_{\rm tot}}{\delta \psi_1 \delta \bar{\psi}_1} + \frac{\delta^2 S_{\rm tot}}{\delta \bar{\psi}_1 \delta \psi_1}\right) = \frac{M}{2}\left(\frac{\delta \psi_1}{\delta \psi_1} + \frac{\delta \bar{\psi}_1}{\delta \bar{\psi}_1} \right) + \frac{\delta^2 S_{\rm eff}}{\delta \psi_1 \delta \bar{\psi}_1} , \tag{11}$$

which tells us that the quantity (9) is also due to the mass term of the PV field, and the equations $\langle \delta S_{\rm tot}/\delta\psi_1\rangle = \langle \delta S_{\rm tot}/\delta\bar{\psi}_1\rangle = 0$. From a comparison between eq. (10) and the M derivative of $Z[J, A; M, M_0]$, the physics is kept unchanged lowering the scale if the following RG equations are satisfied:

[3] In [6, 8] the external gauge field is a component of a vector superfield.

[4] These terms contribute to the partition function with a non-influential overall factor.

1. RG equation for the effective action

$$\left\langle M\frac{\partial}{\partial M}\left\{S_{\text{eff}}+\frac{1}{2}\ln\left(\frac{M}{M_0}\right)\int_x\left(\frac{\delta\psi_1}{\delta\psi_1}+\frac{\delta\bar{\psi}_1}{\delta\bar{\psi}_1}\right)\right\}\right\rangle$$
$$=-\frac{1}{M}\int_x\left\langle\frac{\delta S_{\text{eff}}}{\delta\psi_1}\frac{\delta S_{\text{eff}}}{\delta\bar{\psi}_1}-\frac{\delta^2 S_{\text{eff}}}{\delta\psi_1\delta\bar{\psi}_1}\right\rangle \tag{12}$$

with the initial condition (4).

2. RG equation for the support f_M

$$M\frac{\partial f_M}{\partial M}=f_M \tag{13}$$

with the initial condition (5), for which the solution is obviously $f_M=M/M_0$.

In fact, it follows the condition of RG invariance:

$$M\frac{\partial}{\partial M}\left\{\exp\left(-\frac{M}{M_0^2}\int_x\bar{\chi}_1\chi_1\right)Z[J,A;M,M_0]\right\}=0\ . \tag{14}$$

Using the initial condition $\lim_{M\to M_0}Z[J,A;M,M_0]=Z[J,A;M_0]$, which is a consequence of (4) and (5), the solution of the last equation is

$$Z[J,A;M_0]=\exp\left\{\left(\frac{1}{M_0}-\frac{M}{M_0^2}\right)\int_x\bar{\chi}_1\chi_1\right\}Z[J,A;M,M_0]\ . \tag{15}$$

The quantity (9) is evaluated by using Fujikawa's path integral approach to the anomalous Ward-Takahashi identities. By the rescaling

$$\begin{aligned}\psi_1&\longrightarrow\psi_1'=e^{\alpha}\psi_1\ ,\\ \bar{\psi}_1&\longrightarrow\bar{\psi}_1'=e^{\alpha}\bar{\psi}_1\ ,\end{aligned} \tag{16}$$

with α function of x, the measure of the functional integral transforms as follows:

$$\mathcal{D}\Psi\longrightarrow\mathcal{D}\Psi'=\mathcal{D}\Psi\ \exp 2\int_x\alpha\mathcal{A}_1=\mathcal{D}\Psi\ \exp\frac{1}{12\pi^2}\int_x\alpha F_{\mu\nu}^2\ . \tag{17}$$

We have used the result quoted in appendix B and the commutative nature of the PV field. The related anomalous Ward-Takahashi identity is obtained by a variational derivative:

$$0=\frac{\delta}{\delta\alpha}Z[J,A;M,M_0]\bigg|_{\alpha=0}=\left\langle\frac{\delta S_{\text{tot}}}{\delta\psi_1}\psi_1+\bar{\psi}_1\frac{\delta S_{\text{tot}}}{\delta\bar{\psi}_1}-2\mathcal{A}_1\right\rangle\ . \tag{18}$$

On the other hand, the identity

$$\int\mathcal{D}\Psi\ \left\{\frac{\delta}{\delta\psi_1}\left(\psi_1 e^{-S_{\text{tot}}}\right)+\frac{\delta}{\delta\bar{\psi}_1}\left(\bar{\psi}_1 e^{-S_{\text{tot}}}\right)\right\}=0 \tag{19}$$

turns out to be

$$\left\langle \frac{\delta\psi_1}{\delta\psi_1} + \frac{\delta\bar{\psi}_1}{\delta\bar{\psi}_1} \right\rangle = \left\langle \frac{\delta S_{\rm tot}}{\delta\psi_1}\psi_1 + \bar{\psi}_1 \frac{\delta S_{\rm tot}}{\delta\bar{\psi}_1} \right\rangle = \langle 2\mathcal{A}_1 \rangle \ . \tag{20}$$

Finally, using the independence of $\mathcal{A}_1$ on the regulator fields ψ_1 and $\bar{\psi}_1$, eq. (12) becomes

$$M \frac{\partial \widetilde{S}_{\rm eff}}{\partial M} = -\frac{1}{M} \int_x \left(\frac{\delta \widetilde{S}_{\rm eff}}{\delta\psi_1} \frac{\delta \widetilde{S}_{\rm eff}}{\delta\bar{\psi}_1} - \frac{\delta^2 \widetilde{S}_{\rm eff}}{\delta\psi_1 \delta\bar{\psi}_1} \right) , \tag{21}$$

where

$$\widetilde{S}_{\rm eff} = S_{\rm eff} + \frac{1}{24\pi^2} \ln\left(\frac{M}{M_0}\right) \int_x F_{\mu\nu}^2 \ . \tag{22}$$

Note that we have dealt with the anomaly equations in the operator form and only after having evaluated the quantity (9) have we left out the quantum expectation value. In other words, following ref. [8], we have passed from the weak to the strong form of Polchinski's equation. This is an important point because only by working with the Wilsonian effective action ($S_{\rm eff}$), can the relevance of the rescaling anomaly for the low energy theory be studied. In fact, while the 1PI effective action is a c-number function of classical fields, $S_{\rm eff}$ is an operator which retains quantum fields that have not been integrated out yet and therefore the correct Jacobian has to be taken into account after a rescaling of the fields[5].

As in ref. [6] we have identified the normal ($\widetilde{S}_{\rm eff}$) and anomalous part of the Wilsonian effective action. It is the latter that is responsible for the rescaling of the electric charge. In fact, the solution $\widetilde{S}_{\rm eff}[A,\Psi|_{\Phi^{\rm PV}=0},\mathcal{J};M,M_0]$ of eq. (21) – in terms of which the low energy physics at the momentum scale $p \sim M' \ll M < M_0$ is given – varies rather slowly:

$$\widetilde{S}_{\rm eff}[A,\Psi|_{\Phi^{\rm PV}=0},\mathcal{J};M,M_0] \simeq \widetilde{S}_{\rm eff}[A,\Psi|_{\Phi^{\rm PV}=0},\mathcal{J};M_0,M_0] + O(1/M,1/M_0) \ . \tag{23}$$

Using (22) and the initial condition (4), we obtain

$$\begin{aligned} S_{\rm eff}[A,\Psi|_{\Phi^{\rm PV}=0},\mathcal{J};M,M_0] \simeq {} & \frac{1}{4}\left(\frac{1}{e_0^2} - \frac{1}{6\pi^2}\ln\frac{M}{M_0}\right)\int_x F_{\mu\nu}^2 \\ & - \int_x \bar{\psi}(i\not{D} - m_0)\psi - \int_x (\bar{\chi}\psi + \bar{\psi}\chi) \ . \end{aligned} \tag{24}$$

If we set $\Phi = 0$, the term on the left-hand side of (24) which has a $F_{\mu\nu}^2$ structure is selected, giving

$$\frac{1}{e^2(M)} = \frac{1}{e_0^2} - \frac{1}{6\pi^2}\ln\frac{M}{M_0} \ , \tag{25}$$

from which the well known result of the one-loop β-function can be obtained.

[5] It could be shown that the anomalous term in eq. (22) is subtracted in the formal transition from $\widetilde{S}_{\rm eff}$ to the generator of connected Green's functions with an infrared mass cut-off M – which is obtained with the integration of eq. (21) – if the latter is correctly normalized. It is a result of the classical nature of its fields.

2 Regularized gauge invariant effective action

In this section we use the Slavnov regularization of gauge theories [11], in the framework of the background field method, to regularize the theory at the one-loop level.

It is a well known fact that, as a consequence of the gauge fixing process, we have to deal with non-gauge invariant quantities in the intermediate stage of the calculation of the S-matrix. The gauge invariance of physical quantities is guaranteed if the renormalization procedure satisfies Slavnov-Taylor identities. However, there is a method that retains a residual gauge invariance so that background Slavnov-Taylor identities are fulfilled automatically. This is the background field method (see for instance [17]–[19]).

First of all, closely following Abbott's paper [17], we shall give a brief presentation of the background field formalism that incorporates the matter. Each field of the theory is considered as a sum of a classical background part $\Phi_i^{\rm B} = \{A_\mu^a, c_{\rm B}^a, \bar{c}_{\rm B}^a, \psi_{\rm B}^f, \bar{\psi}_{\rm B}^f\}$ and a quantum piece $\Phi_i = \{Q_\mu^a, c^a, \bar{c}^a, \psi^f, \bar{\psi}^f\}$, which represents the quantum fluctuation around the background field. Then, using the covariant α background gauge, the Euclidean generating functional of the non-Abelian theory can be written as

$$\begin{aligned}\widetilde{Z}[\mathcal{J},\Phi_{\rm B}] = \int \mathcal{D}\Phi\ \exp\Big\{ &- S_{\rm YM}(A+Q) - \frac{1}{2\alpha g_0^2}\int_x (D_\mu Q_\mu)^a (D_\nu Q_\nu)^a \\ &+ \int_x (\bar{c}_{\rm B}+\bar{c})^a D_\mu^{ab} D_\mu^{bd}(A+Q)(c_{\rm B}+c)^d \\ &+ \int_x (\bar{\psi}_{\rm B}+\bar{\psi})^f \left[i\,\not{D}(A+Q) - m_0^f \right] (\psi_{\rm B}+\psi)^f + \int_x \mathcal{J}_i \Phi_i \Big\} ,\end{aligned} \tag{26}$$

where

$$\begin{aligned}&\mathcal{L}_{\rm YM}(Q) = \frac{1}{4g_0^2} F_{\mu\nu}^a(Q) F_{\mu\nu}^a(Q) \qquad \text{with}\ \ F_{\mu\nu}^a(Q) = \partial_\mu Q_\nu^a - \partial_\nu Q_\mu^a + f^{abc} Q_\mu^b Q_\nu^c ,\\ &D_\mu = D_\mu(A) = \partial_\mu - iA_\mu = \partial_\mu - iA_\mu^a T^a ,\\ &(D_\mu Q_\nu)^a = D_\mu^{ab} Q_\nu^b = \partial_\mu Q_\nu^a + f^{abc} A_\mu^b Q_\nu^c ,\\ &\mathcal{J}_i = \left\{ j_\mu^a, \bar{\eta}^a, -\eta^a, \bar{\chi}^f, -\chi^f \right\} .\end{aligned} \tag{27}$$

Furthermore, the color indices have been suppressed and $f = 1, \ldots, N_f$ is a flavor index, where N_f is the number of quark flavors. The gauge group is of color $SU(N)$ with the Hermitian generators that satisfy the typical relations $[T^a, T^b] = i f^{abc} T^c$ of the Lie algebra and are normalized as follows:

$$\mathrm{Tr}(T^a T^b) = t_2(R)\delta^{ab} . \tag{28}$$

$t_2(R)$ is the Dynkin index of the R representation, for which $t_2(A) = N$ and $t_2(N) = 1/2$ when the adjoint (A) and fundamental (N) representation are considered.

The generating functional in (26) has the remarkable property of being invariant under simultaneous infinitesimal transformations of the background fields

$$
\begin{aligned}
\delta A^a_\mu &= (D_\mu\omega)^a \,, && \\
\delta\psi^f_{\rm B} &= i\omega^a T^a\psi^f_{\rm B} \,, & \delta\bar\psi^f_{\rm B} &= -i\bar\psi^f_{\rm B}\omega^a T^a \,, \\
\delta c^a_{\rm B} &= f^{abc}c^b_{\rm B}\omega^c \,, & \delta\bar c^a_{\rm B} &= f^{abc}\bar c^b_{\rm B}\omega^c \,,
\end{aligned}
\tag{29}
$$

and the sources

$$
\begin{aligned}
\delta j^a_\mu &= f^{abc}j^b_\mu\omega^c \,, && \\
\delta\bar\chi^f &= -i\bar\chi^f\omega^a T^a \,, & \delta\chi^f &= i\omega^a T^a\chi^f \,, \\
\delta\bar\eta^a_{\rm B} &= f^{abc}\bar\eta^b_{\rm B}\omega^c \,, & \delta\eta^a_{\rm B} &= f^{abc}\eta^b_{\rm B}\omega^c \,.
\end{aligned}
\tag{30}
$$

Moreover, its connected part $\widetilde{W}[\mathcal{J},\Phi_{\rm B}] = \ln\widetilde{Z}[\mathcal{J},\Phi_{\rm B}]$ is equal to the background gauge invariant effective action $\widetilde\Gamma[0,\Phi_{\rm B}]$ – for which an equivalence proof of the background field quantization method with the conventional one can be inferred from refs. in [20] – if the sources $\mathcal{J}$ are $\Phi_{\rm B}$-dependent in such a way that a generalized 't Hooft equation

$$
\frac{\delta\widetilde{W}}{\delta\Phi^{\rm B}_i(x)} + \int_y \frac{\delta\mathcal{J}_j(y)}{\delta\Phi^{\rm B}_i(x)}\frac{\delta\widetilde{W}}{\delta\mathcal{J}_j(y)} = -(-1)^{\delta_i}\mathcal{J}_i(x)
\tag{31}
$$

is satisfied. This is demonstrated in appendix A generalizing the equivalence proof of 't Hooft's procedure [18] with that of Abbott's [17], when fermions are incorporated into the theory. We have introduced the fermionic number δ_i such that $(-1)^{\delta_i} = 1$ and $(-1)^{\delta_i} = -1$ for bosonic and fermionic variables respectively[6].

The background gauge invariance sets constraints on the infinities that appear in $\widetilde\Gamma[0,\Phi_{\rm B}]$ (see refs. [17, 21]). They must take the following gauge invariant form[7]

$$
\widetilde\Gamma^\infty_0 = \int_x \Big\{ C_1(F^a_{\mu\nu})^2 + C_2\bar\psi^f_{\rm B}\,\not{D}\psi^f_{\rm B} + C_3\bar\psi^f_{\rm B}\psi^f_{\rm B} + C_4(D_\mu\bar c_{\rm B})^a(D_\mu c_{\rm B})^a \Big\} \,,
\tag{32}
$$

where C_n, with $n = 1,\ldots,4$, are infinite constants and the lower index on the left-hand side means that we are taking the bare quantities on the other side. In terms of renormalized fields $\Phi^{\rm B}_i = Z_i^{-1/2}(\Phi^{\rm B}_0)_i$ the last identity becomes

$$
\widetilde\Gamma^\infty = \int_x \Big\{ C_1 Z_A^{1/2}(F^a_{\mu\nu})^2 + C_2 Z_{\psi_{\rm B}}\bar\psi^f_{\rm B}\,\not{D}\psi^f_{\rm B} + C_3 Z_{\psi_{\rm B}}\bar\psi^f_{\rm B}\psi^f_{\rm B} + C_4 Z_{c_{\rm B}}(D_\mu\bar c_{\rm B})^a(D_\mu c_{\rm B})^a \Big\} \,,
\tag{33}
$$

with $F_{\mu\nu}$ and D_μ that will have the expressions dictated by the gauge invariance if the constant structure and the elements of the Lie algebra are renormalized as follows: $f^{abc} = Z_A^{1/2} f_0^{abc}$

[6] It accounts for the commutation property of the variables involved. For example $\Phi_i\Phi_j = (-1)^{\delta_i\delta_j}\Phi_j\Phi_i$.

[7] For the time being, if we do not indicate the dependence on gauge fields, it will mean that we are considering the background gauge field dependence. For example $F^a_{\mu\nu} \doteq F^a_{\mu\nu}(A)$.

and $T^a = Z_A^{1/2} T_0^a$. These quantities are determined by the Lie algebra relations except for a multiplicative common factor, which is the gauge coupling constant, if the Lie algebra is simple. Thus, the gauge coupling must renormalize as $g = Z_A^{1/2} g_0$, which means that the gauge coupling and background gauge field renormalization are related. In fact, defining $Z_g = g_0/g$, the relation $Z_g = Z_A^{-1/2}$ is obtained, that is to say the β-function originates solely from the background gauge field two-point function [17]. This is the reason why from now on we shall be interested in the gauge invariant effective action $\widetilde{\Gamma}[0, A] = \widetilde{W}[\mathcal{J}[A], A]$, whose path integral representation is deduced from eq. (26) setting $\psi_{\rm B}^f = \bar{\psi}_{\rm B}^f = c_{\rm B}^a = \bar{c}_{\rm B}^a = 0$, with the sources $\mathcal{J}_i[A]$ which are solutions of suitable 't Hooft's equations[8].

With the intention to calculate the one-loop β-function in the next section taking full advantage of the gauge invariance, we regularize the functional $\widetilde{\Gamma}[0, A]$ using the regularization proposed by Slavnov [11] at the one-loop level. It consists of the following two steps. The first one is a gauge invariant generalization of the higher derivatives regularization. In fact, to improve the ultraviolet behavior of propagators, the gauge invariance requires the introduction of covariant instead of ordinary derivatives into the kinetic term of the action [10]. Thus, the Yang-Mills action and the gauge fixing surface (G^a) are replaced by the substitutions

$$S_{\rm YM}(A+Q) \longrightarrow S_{\rm YM}^{n,\Lambda}(A+Q) = \frac{1}{4g_0^2} \int_x \left\{ F_{\mu\nu}^2 + \frac{1}{\Lambda^{2n}} (D^n F_{\mu\nu})^2 \right\} (A+Q) \, , \tag{34}$$

$$G^a = (D_\mu Q_\mu)^a \longrightarrow F_n(D^2/\Lambda^2)(D_\mu Q_\mu)^a \, , \tag{35}$$

where F_n is a polynomial of an order greater than equal to $n/2$ and from now on $V^2 \doteq V^a V^a$.

For the reason mentioned above, $S_{\rm YM}^{n,\Lambda}(A+Q)$ is invariant under the quantum gauge transformation $\delta(A+Q)_\mu^a = \delta Q_\mu^a = D_\mu^{ab}(A+Q)\omega^b$. Moreover, the substitutions (34) and (35) yield a functional $\widetilde{\Gamma}_\Lambda^n[0, A]$ still invariant under the background gauge transformation $\delta A_\mu^a = (D_\mu \omega)^a$. Therefore, the advantages of background field method are retained in the regularized theory. For instance, the identity $Z_g = Z_A^{-1/2}$ remains true in the regularized theory. This is a significant property, which is a result of using the background field approach to the Slavnov regularization.

An inspection of the superficial degree of divergence of Feynman's diagrams, with the classical field A on external lines and quantum fields Q, c, $\bar{c}$, ψ and $\bar{\psi}$ inside loops, tells us that the infinities only appear at the one-loop level if $n \geq 2$ [10, 11, 13, 14] and matter loops are regularized by the conventional PV regularization [16]. The second step concerns the regularization of remaining divergences using the gauge invariant PV procedure extended to Yang-Mills and ghost loops [11].

The one-loop contribution to $\widetilde{\Gamma}_\Lambda^n[0, A]$ is given by the partition function

$$\mathcal{Z}_\Lambda^n[A] = \exp \widetilde{\Gamma}_{n,\Lambda}^{\text{1-loop}}[0, A] = \int \mathcal{D}Q \, \exp \left\{ -\frac{1}{2} \int_{xy} \frac{\delta^2 S_{\rm YM}^{n,\Lambda}}{\delta A_\mu^a(x) \delta A_\nu^b(y)} Q_\mu^a(x) Q_\nu^b(y) \right.$$

[8] They are obtained from eqs. (31) and (84) noting that the condition $\widetilde{\Phi}_i = 0$ is now equivalent to $\overline{\Phi}_i = \delta_{i1} A_\mu^a$ (see the procedure in appendix A).

$$-\frac{1}{2\alpha g_0^2}\int_x \left[F_n(D^2/\Lambda^2)D_\mu Q_\mu\right]^2 \Big\} \det(i\not{D}-m_0^f)\det\left[F_n(D^2/\Lambda^2)D^2\right] \;, \tag{36}$$

whose divergences can be cured compatibly with background gauge invariance adding mass terms to each quantum field. In fact, the functional

$$\begin{aligned}\mathcal{Z}^n_{\Lambda,M_i,\mu_j,m_k}[A] &= \mathcal{Z}^n_\Lambda[A]\prod_{f=1}^{N_f}\prod_{i,j,k=1}^{n_1,n_2,n_3}\det{}^{-\alpha_i/2}\mathcal{Q}(A,M_i,F_n) \\ &\times\det{}^{\beta_j}\left[F_n(D^2/\Lambda^2)D^2-\mu_j^2\right]\det{}^{\gamma_k}(i\not{D}-m_k^f) \;,\end{aligned} \tag{37}$$

where

$$\begin{aligned}\det{}^{-1/2}\mathcal{Q}(A,M,F_n) &= \int \mathcal{D}Q\;\exp\Big\{-\frac{1}{2}\int_{xy}\frac{\delta^2 S^{n,\Lambda}_{\rm YM}}{\delta A^a_\mu(x)\delta A^b_\nu(y)}Q^a_\mu(x)Q^b_\nu(y) \\ &-\frac{1}{2\alpha g_0^2}\int_x\left[F_n(D^2/\Lambda^2)D_\mu Q_\mu\right]^2-\frac{M^2}{2}\int_x Q_\mu^2\Big\} \;,\end{aligned} \tag{38}$$

is gauge invariant and even free of divergences if the PV conditions

$$\begin{aligned}&\sum_{i=0}^{n_1}\alpha_i = 0\;, &&\sum_{j=0}^{n_2}\beta_j = 0\;, &&\sum_{k=0}^{n_3}\gamma_k = 0\;, \\ &\sum_{i=0}^{n_1}\alpha_i M_i^2 = 0\;, &&\sum_{j=0}^{n_2}\beta_j\mu_j^2 = 0\;, &&\sum_{k=0}^{n_3}\gamma_k(m_k^f)^2 = 0\end{aligned} \tag{39}$$

are satisfied. In these equations $\alpha_0 = \beta_0 = \gamma_0 = 1$ and $M_0 = \mu_0 = 0$. Note that, at the one-loop level, there is no need to introduce a pre-regulator and change the PV conditions in order to solve the overlapping divergences problem [13], which is due to subdiagrams that are not regularized by the PV procedure.

The coefficients α_i (β_j and γ_k) must be integers in order that they can be interpreted as the number of PV vector (scalar and spinor) fields of the regularized local Lagrangian, whose masses are M_i (μ_j and m_k^f). In this case the PV procedure amounts to subtract from each kind of loop a sequence of analogous loops, along which massive fields propagate, which transform under the same representation as the homogeneous Lorentz group of the physical field in the former loop. The PV fields corresponding to $\alpha_i < 0$ and $\beta_j > 0$ are of fermionic type and those corresponding to $\gamma_k < 0$ are bosonics. Therefore, they do not satisfy the spin-statistic relation. However, the spin-statistic theorem is not violated because, decoupling from the physical fields when the mass regulators go to infinity, no PV regulator field appears in the asymptotic states. It should be mentioned that, in the regularization scheme we are using, this is true in any $\alpha \neq 0$ gauge. When $\alpha = 0$ the regulator fields do not decouple completely. In fact, the Landau gauge does not give the correct value of the one-loop β-function of the pure Yang-Mills theory as has been shown in ref. [12], which lead the authors to state a no-go

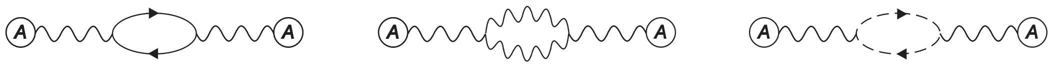

Figure 1: One-loop diagrams with the background gauge field A on external lines. Internal wavy lines are quantum gauge propagators and dashed lines are ghost propagators.

theorem concerning the Slavnov regularization. Even if a minor modification of the scheme exists [13], in which the correctness of the one-loop result on the β-function is guaranteed, for the rest of this paper we shall assume an $\alpha \neq 0$ gauge.

The Slavnov regularization does not specify the PV regulator system. The only reasonable requirements that have to be satisfied in addition to the conditions (39) are the following. The coefficients α_i, β_j and γ_k must be chosen as integers. The variability field of the mass regulators M_i, μ_j and m_k^f has to include infinity, which corresponds to the removal of the PV part of the regularization. One of the different systems of PV regulator fields is sufficient to calculate the one-loop β-function with the RG method. However, it is worth checking the independence of the one-loop β-function on the PV regulator system.

A suitable system for the calculation of the one-loop β-function can be deduced from the relation $Z_g = Z_A^{-1/2}$. From the knowledge of Z_A, for which only the background field two-point function is required, we can determine the β-function. Therefore, no vertex function or tadpole diagram need to be considered, and, at the one-loop level, we only need to regularize the Feynman diagrams in figure 1. The vacuum polarization diagram (a) can be regularized as in section 1. Then, there is only one class of bosonic fields ψ_1^f with mass M in the spinor sector of the PV regulator system. The diagrams (b) and (c) in figure 1 are regularized if the usual PV conditions are satisfied. These conditions can be realized through the introduction of at least two auxiliary masses. In such a case we find

$$\alpha_1 = \frac{M_2^2}{M_1^2 - M_2^2} \, , \qquad \alpha_2 = \frac{M_1^2}{M_2^2 - M_1^2} \tag{40}$$

for PV vector fields and the same for PV scalar fields replacing $\alpha_i|_{i=1,2}$ and $M_i|_{i=1,2}$ with $\beta_j|_{j=1,2}$ and $\mu_j|_{j=1,2}$ respectively. Choosing the integer values $\alpha_1 = \beta_1 = 1$, it follows $\alpha_2 = \beta_2 = -2$. Thus, the PV vectorial sector is composed of one class of bosonic fields $Q^a_{1,\mu}$ of mass M_1 and two mass degenerate classes of fermionic fields $Q^a_{2,\mu}$ and $Q^a_{3,\mu}$ with mass $M_2 = M_1/\sqrt{2}$; the scalar sector of a set of fields $c^a_{1,\mu}$, $c^a_{2,\mu}$ and $c^a_{3,\mu}$ with opposite statistics and masses μ_1 and $\mu_2 = \mu_3 = \mu_1/\sqrt{2}$. Noting that the vector, scalar and spinor loops are regularized separately, we can set $M_2 = \mu_2 = M$.

Another system of PV regulator fields is inferred from the chiral gauge invariant PV regularization proposed by Frolov and Slavnov [22]. Each "sector" is composed of an infinite number of fields with alternating statistics, which corresponds to the choice of $\alpha_i = \beta_i = \gamma_i = (-1)^i$ for $i = \pm 1, \pm 2, \ldots, \pm\infty$. However, a fundamental issue is to define how to sum over the infinite number in order to satisfy the first PV condition. In other words, we have to

define the symbol $\sum_{n=-\infty}^{+\infty}(-1)^n$, which is a divergent series[9]. There are various methods of summing divergent series as part of the theory of divergent series, for which we shall refer to Hardy's book [23]. Then, the series $\sum_{n=0}^{\infty}(-1)^n$ being Cesàro, Abel and Euler summable to 1/2, the following manipulations

$$\begin{aligned} 0 &= \sum_{n=0}^{+\infty}(-1)^n - \sum_{n=0}^{+\infty}(-1)^n = \sum_{n=0}^{+\infty}(-1)^n + \sum_{n=0}^{+\infty}(-1)^{n+1} = \sum_{n=0}^{+\infty}(-1)^n + \sum_{n=1}^{+\infty}(-1)^n \\ &= \sum_{n=0}^{+\infty}(-1)^n + \sum_{n=-\infty}^{-1}(-1)^n = \sum_{n=-\infty}^{+\infty}(-1)^n \end{aligned} \tag{41}$$

are correct. Therefore, the first PV condition is satisfied with respect to the criteria of summability mentioned above.

The physical meaning underlying this mathematical topic is as follows. We attempt to subtract the divergence of the physical sector, which is selected by $i = j = k = 0$, introducing a pair of PV fields with the same statistic. This is equivalent to subtracting the divergence twice, since we are considering $-1+1-1 = -1$. To remove this divergence we need to introduce another pair with the opposite statistic of the former, which yields $+1-1+1-1+1 = +1$. This argument makes clear that it is hopeless trying to regularize the theory by a finite number of PV fields with alternating statistics. Then, an infinite number is introduced giving the possibility to reiterate to infinity the above steps until the divergence is removed [24]. However, in our opinion, it is necessary to assign a specific meaning to the symbol $\sum_{n=-\infty}^{+\infty}(-1)^n$ by making use of the divergent series theory.

The second PV condition can be formally satisfied with a proper system of mass regulators. In fact, using the Leibniz series

$$\begin{aligned} \sum_{n=1}^{+\infty}\frac{1}{n^2} = \frac{\pi^2}{6} \qquad &\longrightarrow \qquad \sum_{n=-\infty}^{+\infty}{}' \frac{1}{n^2} = \frac{\pi^2}{3}\,, \\ \sum_{n=1}^{+\infty}\frac{(-1)^{n-1}}{n^2} = \frac{\pi^2}{12} \qquad &\longrightarrow \qquad \sum_{n=-\infty}^{+\infty}{}' \frac{(-1)^n}{n^2} = -\frac{\pi^2}{6}\,, \end{aligned} \tag{42}$$

where $\sum_n' \doteq \sum_{n\neq 0}$, we obtain

$$\begin{aligned} 0 &= \sum_{n=-\infty}^{+\infty}{}' \frac{(-1)^n}{n^2} + \frac{\pi^2}{6} = \sum_{n=-\infty}^{+\infty}{}' \frac{(-1)^n}{n^2} + \frac{1}{2}\sum_{n=-\infty}^{+\infty}{}' \frac{1}{n^2} \\ &= \sum_{n=-\infty}^{+\infty}{}' \frac{(-1)^n}{n^2}\left[1 + \frac{(-1)^n}{2}\right]\,. \end{aligned} \tag{43}$$

[9]We follow Hardy's criterion [23] to term a divergent series the one that does not converge according to the classical definition of Cauchy.

Then, remembering that each kind of loop is regularized separately, we can set $M_n^2 = \mu_n^2 = M^2 u_n$ and $(m_n^f)^2 = M^2 v_n^f$ with

$$u_n = \frac{1}{n^2}\left[1 + \frac{(-1)^n}{2}\right] , \qquad v_n^f = \frac{1}{n^2}\left[1 + \frac{(-1)^n}{2} + \frac{6(m_0^f)^2}{\pi^2}\right] , \tag{44}$$

to satisfy the second PV conditions. Note that $u_n, v_n^f > 0 \ \forall n \neq 0$. Therefore, choosing $M_n = \mu_n = M\sqrt{u_n}$ and $m_n^f = M(v_n^f)^{1/2}$, the removal of the PV regularization is given by $M \to \infty$.

The proof that this generalized PV regularization works is given in ref. [25] for the pure Yang-Mills theory taking as mass regulator $M_i = M|i|$ and $\mu_j = \mu|j| \ \forall i, j$. We think that it can be extended to include the matter taking $m_k^f = m|k|$ for $k \neq 0$ and even to the background field formalism using the tools developed in [26]. The higher covariant derivatives complicate the Feynman rules and hence make the above proofs a difficult task. However, as will become clearer in the next section, these complications could be avoided in the calculation of the one-loop β-function. In fact, if we had used the relation $Z_g = Z_A^{-1/2}$ from the beginning, we would have had to regularize solely the graphs of figure 1 introducing only PV regulator fields, which does not spoil the gauge invariance of $\widetilde{\Gamma}[0, A]$. Then, using the Feynman rules derived by Abbott in ref. [17], the leading divergence of diagrams (b) and (c) would have been given by[10]

$$\begin{aligned} \int_p p^2 \sum_{n=-\infty}^{+\infty} \frac{(-1)^n}{(p^2 + M^2 n^2)^2} &= -\int_p p^2 \frac{\partial}{\partial p^2} \sum_{n=-\infty}^{+\infty} \frac{(-1)^n}{p^2 + M^2 n^2} \\ &= -\int_p p^2 \frac{\partial}{\partial p^2} \frac{\pi}{Mp} \frac{1}{\sinh(\pi p/M)} , \end{aligned} \tag{45}$$

which prove their finiteness for finite M. The calculation performed in the next section is a first step towards using the Slavnov regularization in the RG context.

3 One-loop β-function of QCD

Due to the large number of fields involved, we need a more concise notation than the one adopted in section 1. All fields and sources are collected in the column vectors

$$\mathbf{\Psi} = \begin{pmatrix} \mathbf{\Phi} \\ \mathbf{\Phi}^{\mathrm{PV}} \end{pmatrix} , \qquad \mathbf{J} = \begin{pmatrix} \mathcal{J} \\ \mathcal{J}^{\mathrm{PV}} \end{pmatrix} , \tag{46}$$

[10] As usual, the diagram (a) could be regularized using one bosonic PV field.

with

$$\begin{pmatrix} Q^a_{i,\mu}/\sqrt{2} \\ Q^a_{i,\mu}/\sqrt{2} \\ c^a_j \\ \bar{c}^a_j \\ \psi^f_k \\ (\bar{\psi}^f_k)^T \end{pmatrix} = \begin{cases} \mathbf{\Phi} & \text{for } \ i=j=k=0 \\ & \\ \mathbf{\Phi}^{\rm PV} & \text{for } \ i,j,k\neq 0 \end{cases} \tag{47}$$

and, using the fermionic number introduced in the previous section,

$$\begin{pmatrix} j^a_{i,\mu}/\sqrt{2} \\ j^a_{i,\mu}/\sqrt{2} \\ \bar{\eta}^a_j \\ (-1)^{\delta_j}\eta^a_j \\ (\bar{\chi}^f)^T_k \\ (-1)^{\delta_k}\chi^f_k \end{pmatrix} = \begin{cases} \mathcal{J} & \text{for } \ i=j=k=0 \\ & \\ \mathcal{J}^{\rm PV} & \text{for } \ i,j,k\neq 0 \end{cases} . \tag{48}$$

We have considered $\bar{\psi}$ a row vector following the Dirac formalism and the vectorial sector has been doubled to treat it as the scalar and spinor sector. This does not mean that the respective measure in the integral functional that will be considered below doubles, i.e.

$$\mathcal{D}\Psi \equiv \prod_{i,j,k,f} \mathcal{D}Q_i \mathcal{D}c_j \mathcal{D}\bar{c}_j \mathcal{D}\psi^f_k \mathcal{D}\bar{\psi}^f_k \, . \tag{49}$$

The generator functional of QCD, regularized according to the Slavnov regularization in the framework of the background field method, is the following:

$$\begin{aligned} \widetilde{Z}[\mathcal{J}(A), A, \mathcal{J}^{\rm PV}; M_0, \Lambda_0] &= \exp \widetilde{\Gamma}[0, A, \mathcal{J}^{\rm PV}; M_0, \Lambda_0] \\ &= \int \mathcal{D}\Psi \ \exp\Big\{ -\frac{1}{2}(\Phi^{\rm PV}, \mathcal{M}_0 \Phi^{\rm PV}) \\ &\quad -S_{\rm int}[\mathcal{J}(A), A, \Psi; M_0, \Lambda_0] + (\mathcal{J}^{\rm PV}, \Phi^{\rm PV})\Big\} \, . \end{aligned} \tag{50}$$

The notation has been misused in calling

$$\begin{aligned} & S^{n,\Lambda_0}_{\rm YM}(A+Q) + \frac{1}{2}\int_{xy} \frac{\delta^2 S^{n,\Lambda_0}_{\rm YM}}{\delta A^a_\mu(x)\delta A^b_\nu(y)} Q^a_{i,\mu}(x) Q^b_{i,\nu}(y)\Big|_{i\neq 0} + \frac{1}{2\alpha g_0^2}\int_x \left[F_n(D^2/\Lambda_0^2) D_\mu Q_{i,\mu}\right]^2 \\ & - \int_x \bar{c}_j F_n(D^2/\Lambda_0^2) D^2 c_j + i\int_x \bar{c} F_n(D^2/\Lambda_0^2) D_\mu Q_\mu c \\ & - \int_x \bar{\psi}^f_k i \not{D} \psi^f_k - \int_x \bar{\psi}^f (\not{Q} - m^f_0)\psi^f - (\mathcal{J}(A), \Phi) \end{aligned} \tag{51}$$

the interaction action $S_{\rm int}[\mathcal{J}(A), A, \Psi; M_0, \Lambda_0]$, and

$$(\Psi, \mathcal{A}\Psi) = \int_x \mathbf{\Psi}^T \mathcal{A} \mathbf{\Psi} = \int_x \Psi^T_\alpha \mathcal{A}_{\alpha\beta} \Psi_\beta \tag{52}$$

denotes the inner product in the space spanned by the vector $\mathbf{\Psi}$, where $\mathcal{A}$ is a generic matrix. Obviously, $\frac{1}{2}(\Phi^{\rm PV}, \mathcal{M}_0 \Phi^{\rm PV})$ is an inner product in the PV subspace. The mass matrix is

$$\mathcal{M} = \begin{pmatrix} 0 & (-1)^{\delta_i} M_i^2 & 0 & 0 & 0 & 0 \\ M_i^2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & (-1)^{\delta_j} \mu_j^2 & 0 & 0 \\ 0 & 0 & \mu_j^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & (-1)^{\delta_k} m_k^f \\ 0 & 0 & 0 & 0 & m_k^f & 0 \end{pmatrix} , \tag{53}$$

with $i, j, k \neq 0$. The matrix depends on the system of PV regulator fields. The system described in the previous section with a finite number of fields results in the (14×14) matrix whose elements are given by setting $M_2 = M_3 = \mu_2 = \mu_3 = m_1^f = M$, $M_1 = \mu_1 = \sqrt{2}M$, $\delta_i|_{i=1} = \delta_j|_{j=2,3} = 0$, $\delta_i|_{i=2,3} = \delta_j|_{j=1} = 1$ and $\delta_k|_{k=1} = 0$. The one with an infinite number results in the $(\infty \times \infty)$ matrix whose elements are given by setting $\delta_i|_{i={\rm even}} = 0$, $\delta_i|_{i={\rm odd}} = 1$, $\delta_j, \delta_k|_{j,k={\rm even}} = 1$, $\delta_j, \delta_k|_{j,k={\rm odd}} = 0$ and $M_n = \mu_n = M\sqrt{u_n}$, $m_n^f = M(v_n^f)^{1/2}$ or $M_i = \mu_i = m_i^f = M|i|$. The matrix $\mathcal{M}_0$ is $\mathcal{M}$ to the scale M_0, namely $\mathcal{M}_0 \equiv \mathcal{M}(M \to M_0)$.

As in section 1 we vary the mass parameter M_0 to a lower value M while keeping the physics unchanged. In other words, we look for the RG equations, with the initial conditions

$$\lim_{M \to M_0} S_{\rm eff}[\mathcal{J}(A), A, \Psi; M, M_0, \Lambda_0] = S_{\rm int}[\mathcal{J}(A), A, \Psi; M_0, \Lambda_0] \,, \tag{54}$$

$$\lim_{M \to M_0} \tilde{\mathcal{J}}^{\rm PV} = \mathcal{J}^{\rm PV} \,, \tag{55}$$

that have to satisfy $S_{\rm eff}$ and $\tilde{\mathcal{J}}^{\rm PV}$ in order that

$$\begin{aligned} \widetilde{Z}[\mathcal{J}(A), A, \tilde{\mathcal{J}}^{\rm PV}; M, M_0, \Lambda_0] &= \int \mathcal{D}\Psi \, \exp\Big\{ -\frac{1}{2}(\Phi^{\rm PV}, \mathcal{M}\Phi^{\rm PV}) \\ &\qquad - S_{\rm eff}[\mathcal{J}(A), A, \Psi; M, M_0, \Lambda_0] + (\tilde{\mathcal{J}}^{\rm PV}, \Phi^{\rm PV}) \Big\} \\ &\equiv \int \mathcal{D}\Psi \, \exp(-S_{\rm tot}) \end{aligned} \tag{56}$$

is equal to $\widetilde{Z}[\mathcal{J}(A), A, \mathcal{J}^{\rm PV}; M_0, \Lambda_0]$ except for a tree level two-point function. Closely following the Abelian case, by means of the Polchinski identity

$$0 = \int_x (-1)^{\delta_\alpha} M \frac{\partial \mathcal{M}^{-1}_{\alpha\beta}}{\partial M} \int \mathcal{D}\Psi \, \frac{\delta}{\delta \Phi^{\rm PV}_\alpha} \Big\{ \mathcal{M}_{\beta\gamma} \Phi^{\rm PV}_\gamma + \frac{1}{2} \frac{\delta}{\delta (\Phi^{\rm PV})^T_\beta} \Big\} \exp(-S_{\rm tot}) \tag{57}$$

we obtain

$$0 = \int_x \left\langle (-1)^{\delta_\alpha} M \frac{\partial \mathcal{M}^{-1}_{\alpha\beta}}{\partial M} \mathcal{M}_{\beta\gamma} \frac{\delta \Phi^{\rm PV}_\gamma}{\delta \Phi^{\rm PV}_\alpha} - (-1)^{\delta_\alpha} M \frac{\partial \mathcal{M}^{-1}_{\alpha\beta}}{\partial M} \mathcal{M}_{\beta\gamma} \frac{\delta S_{\rm tot}}{\delta \Phi^{\rm PV}_\alpha} \Phi^{\rm PV}_\gamma \right.$$
$$\left. + \frac{(-1)^{\delta_\alpha}}{2} M \frac{\partial \mathcal{M}^{-1}_{\alpha\beta}}{\partial M} \left(\frac{\delta S_{\rm tot}}{\delta \Phi^{\rm PV}_\alpha} \frac{\delta S_{\rm tot}}{\delta (\Phi^{\rm PV})^T_\beta} - \frac{\delta^2 S_{\rm tot}}{\delta \Phi^{\rm PV}_\alpha \delta (\Phi^{\rm PV})^T_\beta} \right) \right\rangle . \tag{58}$$

To go further we need the following properties of the mass matrix

$$\mathcal{M}_{\alpha\beta} = (-1)^{\delta_\alpha \delta_\beta} \mathcal{M}_{\beta\alpha} = (-1)^{\delta_\alpha} \mathcal{M}_{\beta\alpha} = (-1)^{\delta_\beta} \mathcal{M}_{\beta\alpha} . \tag{59}$$

In this paper the fermionic number is never summed over the repeated indices. For example, $(-1)^{\delta_\rho} \mathcal{M}_{\rho\alpha} \mathcal{M}^{-1}_{\rho\beta} \doteq (-1)^{\delta_1} \mathcal{M}_{1\alpha} \mathcal{M}^{-1}_{1\beta} + (-1)^{\delta_2} \mathcal{M}_{2\alpha} \mathcal{M}^{-1}_{2\beta} + \ldots = \delta_{\alpha\beta}$.

Then, substituting $S_{\rm tot}$ in eq. (58) for the expression defined in (56) and using the equation $\langle \delta S_{\rm tot} / \delta \Phi^{\rm PV}_\alpha \rangle = 0$, we get

$$0 = \left\langle \frac{1}{2} \left(\Phi^{\rm PV}, M \frac{\partial \mathcal{M}}{\partial M} \Phi^{\rm PV} \right) - \frac{1}{2} \int_x (-1)^{\delta_\alpha} \mathcal{M}^{-1}_{\alpha\beta} M \frac{\partial \mathcal{M}_{\beta\gamma}}{\partial M} \frac{\delta \Phi^{\rm PV}_\gamma}{\delta \Phi^{\rm PV}_\alpha} \right.$$
$$+ \frac{(-1)^{\delta_\alpha}}{2} M \frac{\partial \mathcal{M}^{-1}_{\alpha\beta}}{\partial M} \int_x \left(\frac{\delta S_{\rm eff}}{\delta \Phi^{\rm PV}_\alpha} \frac{\delta S_{\rm eff}}{\delta (\Phi^{\rm PV})^T_\beta} - \frac{\delta^2 S_{\rm eff}}{\delta \Phi^{\rm PV}_\alpha \delta (\Phi^{\rm PV})^T_\beta} \right)$$
$$\left. - \left(M \frac{\partial \mathcal{M}}{\partial M} \mathcal{M}^{-1} \tilde{\mathcal{J}}^{\rm PV}, \Phi^{\rm PV} \right) - \frac{1}{2} \left(\tilde{\mathcal{J}}^{\rm PV}, (-1)^{\delta} M \frac{\partial \mathcal{M}^{-1}}{\partial M} \tilde{\mathcal{J}}^{\rm PV} \right) \right\rangle . \tag{60}$$

It is an easy task to check the M independence of the matrix $\mathcal{M}^{-1} M \partial \mathcal{M} / \partial M$ for all PV systems considered before. In fact, it turns out to be

$$\mathcal{M}^{-1} M \frac{\partial \mathcal{M}}{\partial M} = \begin{pmatrix} 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \equiv \mathbf{\Delta} . \tag{61}$$

If the following RG equations are satisfied:

1. RG equation for the effective action

$$\left\langle M \frac{\partial}{\partial M} \left\{ S_{\rm eff} + \frac{1}{2} \ln \left(\frac{M}{M_0} \right) \int_x (-1)^{\delta_\alpha} \Delta_{\alpha\beta} \frac{\delta \Phi^{\rm PV}_\beta}{\delta \Phi^{\rm PV}_\alpha} \right\} \right\rangle$$
$$= \frac{(-1)^{\delta_\alpha}}{2} M \frac{\partial \mathcal{M}^{-1}_{\alpha\beta}}{\partial M} \int_x \left\langle \frac{\delta S_{\rm eff}}{\delta \Phi^{\rm PV}_\alpha} \frac{\delta S_{\rm eff}}{\delta (\Phi^{\rm PV})^T_\beta} - \frac{\delta^2 S_{\rm eff}}{\delta \Phi^{\rm PV}_\alpha \delta (\Phi^{\rm PV})^T_\beta} \right\rangle \tag{62}$$

with the initial condition (54).

2. RG equations for the sources

$$M\frac{\partial\mathcal{M}}{\partial M}\mathcal{M}^{-1}\tilde{\mathcal{J}}^{\text{PV}} = M\frac{\partial\tilde{\mathcal{J}}^{\text{PV}}}{\partial M} \tag{63}$$

with the initial condition (55),

the condition of RG invariance

$$M\frac{\partial}{\partial M}\left\{\widetilde{Z}_M \exp\frac{1}{2}\int_{M_0}^{M} dM'\left(\tilde{\mathcal{J}}^{\text{PV}}, (-1)^{\delta}M\frac{\partial\mathcal{M}^{-1}}{\partial M}\tilde{\mathcal{J}}^{\text{PV}}\right)(M\to M')\right\} = 0 \tag{64}$$

is established. In eq. (64) and from now on $\widetilde{Z}_M \doteq \widetilde{Z}[\mathcal{J}(A), A, \tilde{\mathcal{J}}^{\text{PV}}; M, M_0, \Lambda_0]$.

The vacuum energy of the PV regulator fields is deduced from the anomalous Ward-Takahashi identity related to the infinitesimal rescaling

$$\delta\mathbf{\Phi}^{\text{PV}} = \delta\alpha\mathbf{\Delta}\mathbf{\Phi}^{\text{PV}}\ . \tag{65}$$

In fact, the measure of the functional integral transforms into $\mathcal{D}\Psi' = J\mathcal{D}\Psi$ and hence

$$0 = \frac{\delta\widetilde{Z}_M}{\delta(\delta\alpha)}\bigg|_{\delta\alpha=0} = \left\langle \Delta_{\alpha\beta}\Phi^{\text{PV}}_{\beta}\frac{\delta S_{\text{tot}}}{\delta\Phi^{\text{PV}}_{\alpha}} - \frac{\delta\ln J}{\delta(\delta\alpha)}\right\rangle\ , \tag{66}$$

from which, using the identity

$$\int\mathcal{D}\Psi\ \frac{\delta}{\delta\Phi^{\text{PV}}_{\alpha}}\left\{(-1)^{\delta_\alpha}\Delta_{\alpha\beta}\Phi^{\text{PV}}_{\beta}e^{-S_{\text{tot}}}\right\}\ , \tag{67}$$

we obtain

$$\left\langle(-1)^{\delta_\alpha}\Delta_{\alpha\beta}\frac{\delta\Phi^{\text{PV}}_{\beta}}{\delta\Phi^{\text{PV}}_{\alpha}}\right\rangle = \left\langle\Delta_{\alpha\beta}\Phi^{\text{PV}}_{\beta}\frac{\delta S_{\text{tot}}}{\delta\Phi^{\text{PV}}_{\alpha}}\right\rangle = \left\langle\frac{\delta\ln J}{\delta(\delta\alpha)}\right\rangle\ . \tag{68}$$

As shown in appendix B the Jacobian J does not depend on Φ^{PV}. Therefore, taking the strong form of Polchinski's equation, one gets

$$M\frac{\partial\widetilde{S}_{\text{eff}}}{\partial M} = \frac{(-1)^{\delta_\alpha}}{2}M\frac{\partial\mathcal{M}^{-1}_{\alpha\beta}}{\partial M}\int_x\left(\frac{\delta\widetilde{S}_{\text{eff}}}{\delta\Phi^{\text{PV}}_{\alpha}}\frac{\delta\widetilde{S}_{\text{eff}}}{\delta(\Phi^{\text{PV}})^T_{\beta}} - \frac{\delta^2\widetilde{S}_{\text{eff}}}{\delta\Phi^{\text{PV}}_{\alpha}\delta(\Phi^{\text{PV}})^T_{\beta}}\right)\ , \tag{69}$$

where

$$\widetilde{S}_{\text{eff}} = S_{\text{eff}} + \frac{1}{2}\ln\left(\frac{M}{M_0}\right)\int_x\frac{\delta\ln J}{\delta(\delta\alpha)}\ . \tag{70}$$

Now, we want to show the equality of $\ln J$ for all PV regulator systems considered above and that it has the correct value to achieve the one-loop β-function. We start with the system having a finite number of fields. The vector and scalar fields are in the adjoint representation of the $SU(N)$ gauge group; the spinor fields in the fundamental one. Then, using the results quoted in appendix B and the proper statistic of the fields involved, we obtain

$$\begin{aligned}\ln J &= \int_x\delta\alpha\left(2\mathcal{A}_3 - 2\mathcal{A}_3 - 2\mathcal{A}_3 - 4\mathcal{A}_2 + 4\mathcal{A}_2 + 4\mathcal{A}_2 + 2N_f\mathcal{A}_1\right)\\ &= \int_x\delta\alpha\left\{-\frac{11}{48\pi^2}t_2(A) + \frac{N_f}{12\pi^2}t_2(N)\right\}F^2_{\mu\nu}\ .\end{aligned} \tag{71}$$

The analysis of the system with an infinite number of fields requires a clarification. We have satisfied the first PV conditions by making use of some summability criteria of the divergent series theory. It corresponds to assign a fixed order to the infinite products in the functional measure, as can be made clear looking at eq. (37) and thinking how the PV conditions (39) come out. Taking as example the spinor sector, if

$$\det(i\,\not{D}-m_0^f)\prod_k \det{}^{\gamma_k}(i\,\not{D}-m_k^f) \qquad \longrightarrow \qquad \prod_{k=-\infty}^{+\infty}\det{}^{(-1)^k}(i\,\not{D}-m_k^f)$$

$$= \int \prod_{k=-\infty}^{+\infty}\mathcal{D}\psi_k^f\mathcal{D}\bar{\psi}_k^f \ \exp\sum_{k=-\infty}^{+\infty}\int_x \bar{\psi}_k^f(i\,\not{D}-m_k^f)\psi_k^f \ , \tag{72}$$

the first PV condition becomes $\sum_{k=-\infty}^{+\infty}(-1)^k=0$. Thus, the measure functional is

$$\mathcal{D}\Psi=\prod_{i=-\infty}^{+\infty}\mathcal{D}Q_i\prod_{j=-\infty}^{+\infty}\mathcal{D}c_j\mathcal{D}\bar{c}_j\prod_{f=1}^{N_f}\prod_{k=-\infty}^{+\infty}\mathcal{D}\psi_k^f\mathcal{D}\bar{\psi}_k^f \ , \tag{73}$$

and hence

$$\ln J=\int_x \delta\alpha\Big\{2\mathcal{A}_3\sum_{n=-\infty}^{+\infty}{}'\ (-1)^n-4\mathcal{A}_2\sum_{n=-\infty}^{+\infty}{}'\ (-1)^n-2N_f\mathcal{A}_1\sum_{n=-\infty}^{+\infty}{}'\ (-1)^n\Big\} \ . \tag{74}$$

If the series $a_0+a_1+\dots$ is Cesàro, Abel and Euler summable to s then $a_1+a_2+\dots$ is even Cesàro, Abel and Euler summable to $s-a_0$ [23]. Therefore, from the eq. (41) follows $\sum_{n=-\infty}^{+\infty}{}'\ (-1)^n=-1$ providing the same value (71).

As in section 1, the anomalous part of the effective action gives the variation of the gauge coupling constant with the scale. In fact, at low energy,

$$\widetilde{S}_{\text{eff}}[\mathcal{J}(A),A,\Psi|_{\Phi^{\text{PV}}=0};M,M_0,\Lambda_0]\simeq\widetilde{S}_{\text{eff}}[\mathcal{J}(A),A,\Psi|_{\Phi^{\text{PV}}=0};M_0,M_0,\Lambda_0]$$
$$+O(1/M,1/M_0) \ , \tag{75}$$

which yields

$$S_{\text{eff}}[\mathcal{J}(A),A,\Psi|_{\Phi^{\text{PV}}=0};M,M_0,\Lambda_0]\simeq\frac{1}{4}\left\{\frac{1}{g_0^2}+\left[\frac{11}{24\pi^2}t_2(A)-\frac{N_f}{6\pi^2}t_2(N)\right]\ln\frac{M}{M_0}\right\}F_{\mu\nu}^2$$
$$+\cdots \ . \tag{76}$$

$+\cdots$ are terms that do not change under the considered RG flow. Then, we obtain

$$\frac{1}{g^2(M)}=\frac{1}{g_0^2}+\left[\frac{11}{24\pi^2}t_2(A)-\frac{N_f}{6\pi^2}t_2(N)\right]\ln\frac{M}{M_0} \ , \tag{77}$$

and hence the well known result of the one-loop β-function of QCD

$$\beta(g)=g^3\left[\frac{N_f}{12\pi^2}t_2(N)-\frac{11}{48\pi^2}t_2(A)\right] \ . \tag{78}$$

4 Conclusions

Working with the path integral representation of the gauge invariant effective action $\widetilde{\Gamma}[0, A]$ regularized according to the Slavnov regularization, we have given a simple non-diagrammatic RG evaluation of the one-loop β-function in QCD. A significant aspect of the calculation is its compatibility with the gauge invariance. In two respects this classical symmetry is lost in the process of quantization: the regulator may violate the symmetry and the gauge fixing hides the underlying gauge invariance of the theory. In this paper we have shown the advantages of maintaining a manifest background gauge invariance by using a regulator that even regularizes the divergences in a gauge invariant manner.

Another non-diagrammatic one-loop calculation has been worked out by Fujikawa in ref. [29], which is based on a relation between the Weyl anomaly and the β-function. However, our calculation being based on the RG method, it appears to be easier to understand the anomalous origin of the one-loop β-function in terms of scaling of effective Lagrangians.

The method is related to one-loop but could be extended to more then one-loop if we managed to apply the regularization to every loop and knew a way to regulate the Jacobian like the theory. It could also be applied in its simpler form to get the exact β-functions of supersymmetric gauge theories if we were able to give the supersymmetric extension of the regularization. In fact, according to the non-renormalization theorem, the irrelevant operators in the Jacobian, which are D-terms, can be set to zero with no change in the relevant coupling appearing in the F-term of the Jacobian. Furthermore, there are suggestions that a supersymmetric as well as gauge invariant regularization exists [30]. In particular, from West's paper there appears to be a close relation to the regularization scheme adopted in this paper because of the preservation of the background and quantum gauge symmetry in addition to the supersymmetry. Therefore, following our method, the Jacobian could be regularized by hand as in Fujikawa's approach to achieve the exact one-loop running of the holomorphic coupling. This is an important point for the following reasons. The regularization scheme mentioned above could be used as an alternative to the Arkani-Hamed and Murayama regularization [9] when the proof of finiteness appears to be scarce. Unlike the latter regularization, the former is not limited to specific supersymmetric models.

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A 't Hooft's equation

The Legendre transformation of $\widetilde{W}[\mathcal{J},\Phi_{\rm B}]$, assuming that $\widetilde{\Phi}_i = \delta\widetilde{W}/\delta\mathcal{J}_i$ yields an implicit functional equation $\widetilde{\Phi}_i = \widetilde{\Phi}_i[\mathcal{J},\Phi_{\rm B}]$ uniquely solvable with respect to $\mathcal{J}$, is defined as follows:

$$\widetilde{\Gamma}[\widetilde{\Phi},\Phi_{\rm B}] = \widetilde{W}[\mathcal{J}[\widetilde{\Phi},\Phi_{\rm B}],\Phi_{\rm B}] - \int_x \mathcal{J}_i[\widetilde{\Phi},\Phi_{\rm B}]\widetilde{\Phi}_i \ . \tag{79}$$

In this appendix we shall show that $\widetilde{\Gamma}[0,\Phi_{\rm B}] = \widetilde{W}[\mathcal{J}[\Phi_{\rm B}],\Phi_{\rm B}]$ being $\mathcal{J}[\Phi_{\rm B}]$ a functional that satisfies the 't Hooft equation (31).

The change of variables $\Phi \to \Phi - \Phi_{\rm B}$ in the functional integral of eq. (26) results in

$$\widetilde{W}[\mathcal{J},\Phi_{\rm B}] = W[\mathcal{J}] - \int_x \mathcal{J}_i\Phi_i^{\rm B} \ , \tag{80}$$

with $W[\mathcal{J}]$ the conventional generating functional of connected Green's functions evaluated using the gauge fixing surface $G^a = \partial_\mu(Q-A)^a_\mu + f^{abc}A^a_\mu Q^a_\nu$. From the eq. (80) the identities $\widetilde{\Phi}_i = \overline{\Phi}_i - \Phi_i^{\rm B}$ are obtained, which show that the conditions $\widetilde{\Phi}_i = 0$ are equivalent to $\overline{\Phi}_i = \Phi_i^{\rm B}$, where $\overline{\Phi}_i = \delta W/\delta\mathcal{J}_i$. Thus, differentiating $W[\mathcal{J}]$ with respect to $\mathcal{J}$, we must take into account the dependence on $\mathcal{J}$ that is due to the dependence of the gauge fixing term on the background gauge field:

$$\frac{dW}{d\mathcal{J}_i(x)} = \frac{\delta W}{\delta\mathcal{J}_i(x)} + \int_y \frac{\delta A^b_\nu(y)}{\delta\mathcal{J}_i(x)}\frac{\delta W}{\delta A^b_\nu(y)} = \Phi_i^{\rm B}(x) \ . \tag{81}$$

Note that we have distinguished a total from a partial functional derivative with the notations $d/d\mathcal{J}$ and $\delta/\delta\mathcal{J}$ respectively. The conditions $\overline{\Phi}_i = \Phi_i^{\rm B}$ also give to $\mathcal{J}_i$ a dependence on $\Phi_{\rm B}$. Then, considering that the only explicit $\Phi_{\rm B}$-dependence of W is on background gauge fields, we obtain, by making use of eqs. (81),

$$\frac{dW}{d\Phi_i^{\rm B}(x)} = \delta_{i1}\frac{\delta W}{\delta A^a_\mu(x)} + \int_y \frac{\delta\mathcal{J}_j(y)}{\delta\Phi_i^{\rm B}(x)}\frac{\delta W}{\delta\mathcal{J}_j(y)} = \int_y \frac{\delta\mathcal{J}_j(y)}{\delta\Phi_i^{\rm B}(x)}\Phi_j^{\rm B}(y) \ . \tag{82}$$

Finally, we get the 't Hooft equation using eqs. (80), (82) and the fermionic number.

As 't Hooft suggests [18], there is no need to compute $\mathcal{J}_i[\Phi_{\rm B}]$. Nevertheless, the class of solutions may be restricted by the condition that the sources $\mathcal{J}_i[\Phi_{\rm B}]$ transform like (30) when the background fields $\Phi_i^{\rm B}$ undergo the transformations (29). Then $\widetilde{W}[\mathcal{J}[\Phi_{\rm B}],\Phi_{\rm B}]$ becomes a gauge invariant functional of $\Phi_{\rm B}$ and hence

$$\begin{aligned} 0 &= -\int_x \delta\Phi_i^{\rm B}\frac{d\widetilde{W}}{d\Phi_i^{\rm B}} = (-1)^{\delta_i}\int_x \delta\Phi_i^{\rm B}\mathcal{J}_i \\ &= \int_x \{(D_\mu\omega)^a j^a_\mu + i\omega^a\bar{\chi}^f T^a\psi^f_{\rm B} - i\omega^a\bar{\psi}^f_{\rm B}T^a\chi^f + f^{abc}(\bar{\eta}^a c^b_{\rm B}\omega^c + \bar{c}^b_{\rm B}\omega^c\eta^a)\} \\ &= \int_x \omega^a\{-(D_\mu j_\mu)^a + i\bar{\chi}^f T^a\psi^f_{\rm B} - i\bar{\psi}^f_{\rm B}T^a\chi^f + f^{abc}(\bar{\eta}^b c^c_{\rm B} + \bar{c}^c_{\rm B}\eta^b)\} \ , \end{aligned} \tag{83}$$

from which we obtain

$$(D_\mu j_\mu)^a = i(\bar{\chi}^f T^a\psi^f_{\rm B} - \bar{\psi}^f_{\rm B}T^a\chi^f) + f^{abc}(\bar{\eta}^b c^c_{\rm B} + \bar{c}^c_{\rm B}\eta^b) \ . \tag{84}$$

B Anomalous Jacobians under rescaling transformations

Following the Fujikawa approach to the anomaly [27, 28], we look for the operators appearing in the equations of motion. They can be inferred from the quadratic part in quantum variables of the non-regularized action in the Feynman gauge $\alpha = 1$:

$$S_{\mathrm{YM}}(A+Q) - \int_x \bar{\psi}^f \left[i \not{D}(A+Q) - m_0^f \right] \psi^f - \int_x \bar{c} D^2(A+Q) c + \frac{1}{2g_0^2} \int_x (D_\mu Q_\mu)^2$$

$$= -\frac{1}{2g_0^2} \int_x Q_\mu (D^2 \delta_{\mu\nu} - 2i F_{\mu\nu}) Q_\nu - \int_x \bar{\psi}^f (i \not{D} - m_0^f) \psi^f - \int_x \bar{c} D^2 c + \cdots \ , \tag{85}$$

where $+\cdots$ are terms which we are not interested in. Then, under a rescaling transformation like the one in (16)

$$\mathcal{D}\psi \mathcal{D}\bar{\psi} \longrightarrow \mathcal{D}\psi' \mathcal{D}\bar{\psi}' = \mathcal{D}\psi \mathcal{D}\bar{\psi} \ \exp \pm 2 \int_x \alpha \sum_n \varphi_n^\dagger \varphi_n$$

$$\equiv \mathcal{D}\psi \mathcal{D}\bar{\psi} \ \exp \pm 2 \int_x \alpha \mathcal{A}_1 \ , \tag{86}$$

with the plus or minus sign when ψ is a bosonic or fermionic spinor field. In the last equation φ_n is a complete and orthonormal set of eigenfunctions of the Hermitian operator $\not{D}$. Therefore, the function $\mathcal{A}_1$ is divergent. It can be regularized in a gauge invariant manner by smoothly cutting off the contribution of the large eigenvalues and changing the basis vectors φ_n for the plane wave basis as in refs. [27, 28]:

$$\mathcal{A}_1 = \lim_{M\to\infty} M^4 \int_q \mathrm{Tr}\, f\left(q^2 - 2i\frac{(q\cdot D)}{M} + \frac{\not{D}^2}{M^2} \right)$$

$$= \lim_{t\to 0} t^{-4} \int_q \mathrm{Tr}\, f\left(q^2 - 2it(q\cdot D) + t^2 \not{D}^2 \right)$$

$$\equiv \lim_{t\to 0} t^{-4} \int_q \mathrm{Tr}\, F(t) \ , \tag{87}$$

where $\int_q \doteq \int d^4q/(2\pi)^4$. The function $f(s)$ must drop smoothly from 1 to 0 as s goes from 0 to ∞ and $sf'(s) = 0$ at $s = 0$ and $s = \infty$. Developing the matrix function $F(t)$ in power of $t = 1/M$ around $t = 0$, we obtain

$$\mathcal{A}_1 = \lim_{t\to 0} \sum_{n=0}^{4} \frac{1}{n!} t^{n-4} \int_q \mathrm{Tr}\, F^{(n)}(0) + \lim_{t\to 0} \sum_{n=5}^{\infty} \frac{1}{n!} t^{n-4} \int_q \mathrm{Tr}\, F^{(n)}(0) \ . \tag{88}$$

The second term on the right-hand side is the contribution of the irrelevant operators, which is suppressed by negative powers of t. It is zero at the one-loop level since the regulator independent part in the first term contributes with the correct coefficient to the one-loop

β-function, as shown in sections 1 and 3. However, as pointed out in ref. [9], the irrelevant operators in the Jacobian should yield higher loop effects. In fact, according to the RG point of view, there is an infinite number of bare Lagrangians with the same relevant couplings and the same low energy physics, one of which does not have irrelevant couplings. If the Jacobian were regularized like the theory, the operation of setting the irrelevant operators to zero would modify the relevant coupling in the first term of eq. (88) probably providing the higher order corrections to the β-function [9]. Therefore, our method is related to one-loop.

Thus, being $s(t) = q^2 - 2it(q\cdot D) + t^2 \not{D}^2$ a diagonalizable matrix, the conventional rules of derivation can be used under the trace. Then, at the one-loop level

$$\lim_{M\to\infty} M^4 \mathrm{Tr}\, f\left(q^2 - 2i\frac{(q\cdot D)}{M} + \frac{\not{D}^2}{M^2}\right) = \lim_{M\to\infty}\Big\{M^4\mathrm{Tr}\, f(q^2) - 2iM^3 f'(q^2)\mathrm{Tr}(q\cdot D)$$

$$-M^2\left[2f''(q^2)\mathrm{Tr}(q\cdot D)^2 - f'(q^2)\mathrm{Tr}\,\not{D}^2\right]$$

$$+M\left[\frac{4i}{3}f^{(3)}(q^2)\mathrm{Tr}(q\cdot D)^3 - 2if''(q^2)\mathrm{Tr}\,\not{D}^2(q\cdot D)\right]$$

$$+\frac{2}{3}f^{(4)}(q^2)\mathrm{Tr}(q\cdot D)^4 - 2f^{(3)}(q^2)\mathrm{Tr}\,\not{D}^2(q\cdot D)^2 + \frac{1}{2}f''(q^2)\mathrm{Tr}\,\not{D}^4\Big\}\ . \tag{89}$$

Finally, by making use of the integrals

$$\int_q f(q^2) q_{\mu_1}\cdots q_{\mu_n} = 0 \qquad \text{for odd } n,$$
$$\int_q f(q^2) q_\mu q_\nu = \frac{1}{4}\delta_{\mu\nu}\int_q f(q^2) q^2\ , \tag{90}$$
$$\int_q f(q^2) q_\mu q_\nu q_\rho q_\sigma = \frac{1}{24}\left(\delta_{\mu\nu}\delta_{\rho\sigma} + \delta_{\mu\rho}\delta_{\nu\sigma} + \delta_{\mu\sigma}\delta_{\nu\rho}\right)\int_q f(q^2) q^4\ ,$$

and the property of $f(s)$, we obtain

$$\mathcal{A}_1 = \lim_{M\to\infty} M^4 \mathrm{Tr}\int_q f(q^2) + \frac{1}{24\pi^2}\mathrm{Tr}_{\mathrm{G}} F^2_{\mu\nu}\ , \tag{91}$$

where Tr_{G} means a trace only on the gauge group indices. For our purpose, the first term on the right-hand side can be left out as field independent. The same will be done in the following calculation of $\mathcal{A}_2$ and $\mathcal{A}_3$.

The anomalous Jacobians under rescaling transformation of scalar and vector fields are evaluated by the same procedure:

$$\mathcal{D}c\mathcal{D}\bar{c} \longrightarrow \mathcal{D}c'\mathcal{D}\bar{c}' = \mathcal{D}c\mathcal{D}\bar{c}\ \exp\pm 2\int_x \alpha\sum_n \vartheta_n^\dagger\vartheta_n \equiv \mathcal{D}c\mathcal{D}\bar{c}\ \exp\pm 2\int_x \alpha\mathcal{A}_2\ , \tag{92}$$

$$\mathcal{D}Q \longrightarrow \mathcal{D}Q' = \mathcal{D}Q\ \exp\pm\int_x \alpha\sum_n \varrho_n^\dagger\varrho_n \equiv \mathcal{D}Q\ \exp\pm\int_x \alpha\mathcal{A}_3\ , \tag{93}$$

where, according to the eq. (85), ϑ_n and ϱ_n are complete and orthonormal sets of eigenfunctions of the Hermitian operators D^2 and $D^2\delta_{\mu\nu} - 2iF_{\mu\nu}$ respectively. The sign follows the same previous rules. D_μ is an anti-Hermitian operator with respect to the inner product $(c, c) = \int_x c_a^* c^a$ and therefore D^2 is positive semi-definite. Then, suppressing the contribution of large eigenvalues as above, at the one-loop level we get

$$\mathcal{A}_2 = \lim_{M\to\infty} M^4 \int_q \mathrm{Tr}\, f\left(q^2 - 2i\frac{(q\cdot D)}{M} - \frac{D^2}{M^2}\right) = -\frac{1}{192\pi^2}\mathrm{Tr}_{\mathrm{G}} F_{\mu\nu}^2 \ . \tag{94}$$

Repeating the same procedure for the calculation of $\mathcal{A}_3$, we obtain

$$\mathcal{A}_3 = \frac{5}{48\pi^2}\mathrm{Tr}_{\mathrm{G}} F_{\mu\nu}^2 \ . \tag{95}$$

These results have also been worked out in refs. [28, 29] as flat space-time limit of the Weyl anomaly.

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